Last class showed:

Taylor series for $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$

given by

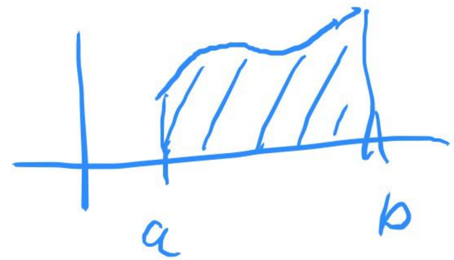
$$\sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

we will show that the value of the Taylor series is indeed equal to $(1+x)^\alpha$ for $|x| < 1$ using an integral version of Taylor's theorem. more later.

$\Rightarrow$ theory of integrals.

Riemann Integral

motivation: calculate area under graph of a positive function



Some definitions:

$S \subset [a,b]$ subset

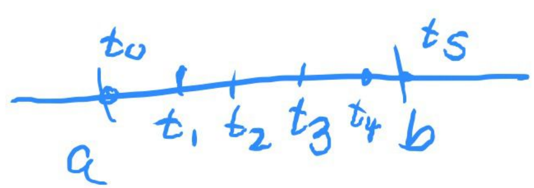
$f: [a,b] \to \mathbb{R}$ bounded function

Define: $M(f,S) = \sup\{f(x), x \in S\}$

$m(f,S) = \inf\{f(x), x \in S\}$

A partition P in $[a,b]$ is a finite collection of points

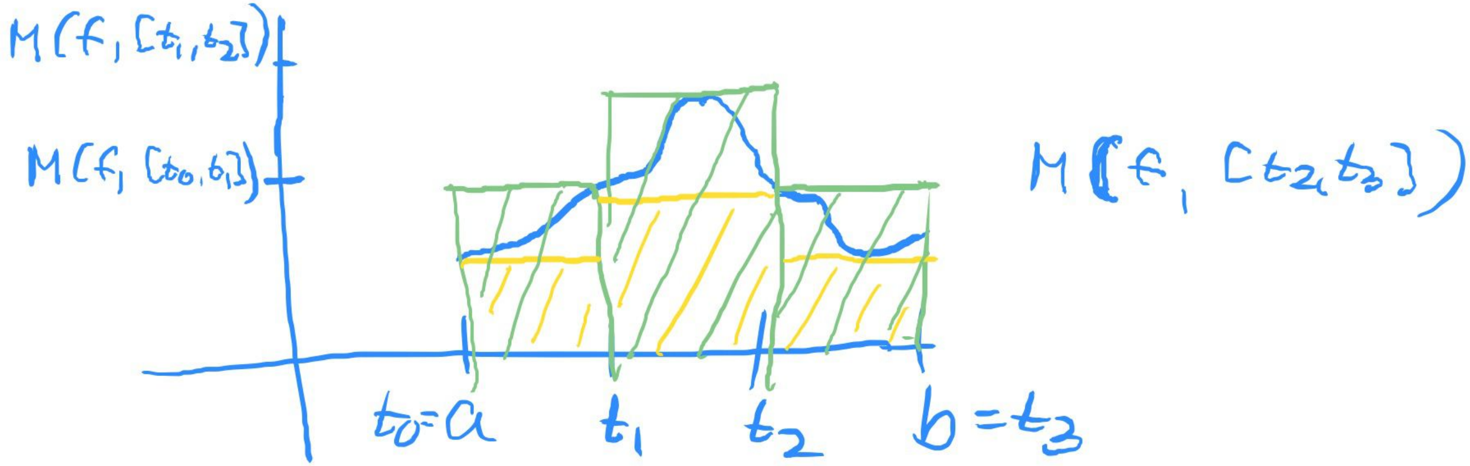
$a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$



<u>Def.</u> Given a bounded function f and a partition P we define the

upper Darboux sum $U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$

lower Darboux sum $L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$



$U(f,P) =$ sum of areas of green rectangles

$L(f,P) =$ " " " " yellow "

Idea: The finer the partitions the closer $U(f,P)$ and $L(f,P)$ will be to the area below graph of f

<u>Def.</u> A function $f:[a,b]\to\mathbb{R}$ is called integrable if

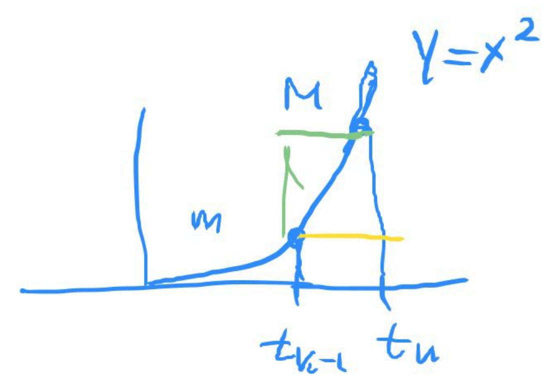
$$U(f) = L(f)$$

where
$$U(f) = \inf\{U(f,P),\ P \text{ a partition of } [a,b]\}$$
$$L(f) = \sup\{L(f,P),\ \text{" " " "}\}$$

<u>Examples</u> ① $f(x)=x^2$, defined on interval $[0,b]$

useful observation: $f(x)$ increasing on $[0,b]$

$$\Rightarrow M(f,[t_{k-1},t_k]) = t_k^2$$
$$m(f,[t_{k-1},t_k]) = t_{k-1}^2$$

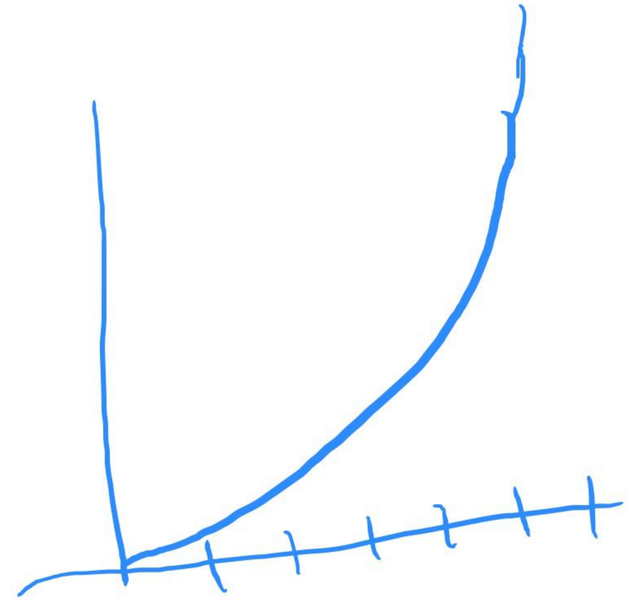


Choose partition $P = \{0 = t_0,\ t_1 = \frac{b}{n},\ t_2 = \frac{2b}{n},\ \dots\ t_{n-1} = \frac{n-1}{n} b,\ t_n = b\}$

i.e. $t_k = \frac{kb}{n}$

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \underbrace{\left(\frac{kb}{n}\right)^2}_{t_k^2} \left(\frac{b}{n}\right)$$



$$= \sum_{k=1}^{n} \frac{k^2 b^2}{n^2} \cdot \frac{b}{n}$$

$$= \frac{b^3}{n^3} \sum_{k=1}^{n} k^2$$

exercise: prove by induction on n:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \boxed{b^3 \cdot \frac{(n+1)(2n+1)}{6n^2}}$$

For $L(f,P)$: get

$$\sum_{k=1}^{n} t_{k-1}^2 \cdot (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \frac{(k-1)^2 b^2}{n^2} \cdot \frac{b}{n} = \frac{b^3}{n^3} \sum_{k=1}^{n-1} k^2$$

$$\ldots = \boxed{\frac{b^3}{6} \cdot \frac{(n-1)(2n-1)}{n^2}}$$

Observe: $\lim_{n\to\infty} U(f,P) = \lim_{n\to\infty} L(f,P) = \frac{b^3}{3}$

$$\left(\text{e.g. for } L(f,P) = \frac{b^3}{6} \cdot \frac{(1-\frac{1}{n})(2-1/n)}{\underbrace{n^2/n^2}_{=1}} \to \frac{b^3}{3} \right)$$

(2) Consider $f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$

defined on $[0,1]$

key observation: if $t_{k-1} < t_k$

$\Rightarrow$ the interval $[t_{k-1}, t_k]$ contains both rational and irrational numbers

$\Rightarrow$ $M(f, [t_{k-1}, t_k]) = 1$ because $\exists$ rational $x \in [t_{k-1}, t_k]$

$m(f, [t_{k-1}, t_k]) = 0$ " " irrational $x \in [t_{k-1}, t_k]$

$\Rightarrow$ If $P = 0 = t_0 < t_1 < \dots < t_n = 1$ is a partition

$$U(f,P) = \sum_{k=1}^{n} \underbrace{M(f, [t_{k-1}, t_k])}_{=1} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k - t_{k-1}) = (1-0) = 1$$

$$L(f,P) = \sum_{k=1} \underbrace{m(f, [t_{k-1}, t_k])}_{=0} (t_k - t_{k-1}) = 0$$

$$\Rightarrow \quad U(f) = \inf\{\underbrace{U(f,P)}_{=1},\ P \text{ a partition of } [0,1]\}$$
$$= 1$$

$$L(f) = \sup\{\underbrace{L(f,P)}_{=0},\ P \text{ a partition of } [0,1]\}$$

$$= 0$$

$$\Rightarrow \quad U(f) \neq L(f)$$

$\Rightarrow$ f is NOT integrable.

For rigorous proofs we need to show:

$$L(f) \leq U(f)$$

It is obviously true that
$$L(f,P) \leq U(f,P) \qquad \text{for a given partition}$$

but it might be conceivable that
$$L(f,P) > U(f,Q) \qquad \text{for different partitions}$$

(not going to happen!)

key <u>Lemma:</u>

Assume P, Q are partitions with $P \subseteq Q$ (i.e. Q is a finer partition)

$$\Rightarrow \quad L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$